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一类延迟线性微分方程两点边值问题的有限体积方法

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摘　要:构造了求解一类延迟二阶线性微分方程的两点边值问题的二阶有限体积法.对求解区间均匀离散,采用线性离散插值方法在每个小区间上对方程进行数值积分,得出相应的数值方法.误差分析显示,在离散 H^1 半范数、L^2 范数以及最大范数下,数值解关于步长都是二阶收敛的.并且,有限的数值结果验证了该数值方法的有效性.

关键词:有限体积法;两点边值问题;均匀网格

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A Finite-volume Method for Delay Differential Equation Two-point Boundary Value Problems

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Abstract: A two-order finite volume method was presented for a class of two-order linear differential equations with delay. Firstly, the interval was divided into a set of small intervals. Then, the finite volume scheme was obtained by integrating the equation on each tiny interval and using the linear discrete interpolation method. Moreover, the errors of the numerical solution, such as discrete H^1 semi-norm error, L^2norm error and maximum norm error, were analyzed, which showed the finite volume scheme is two-order convergent. Finally, numerical results verified the validity of the presented scheme.

Keywords: finite-volume method; two-point boundary value problems; uniform grid

延迟线性微分方程两点边值问题在各个领域都得到了广泛的应用,如流体力学、弹性力学、化学反应、光学等.求解两点边值问题的方法有有限差分法、有限元、有限体积法、打靶法、单调迭代法、拟线性法和广义拟线性法等. 中心差分法[1-2]是将方程整体求解,计算过程简单,但是缺乏高精度.有限元法[3]虽然具有较高的精度,但计算过程复杂.有限体积法[4-5]解微分方程两点边值问题,既有差分法的计算过程简单性,同时具有有限元的高精度性.但鲜有把有限体积法应用到延迟微分方程的情况.在本文中将利用这种方法来研究一类延迟线性微分方程两点边值问题. 打靶法[6-8]把边值问题转化为一族初值问题,进一步转化为非线性方程组来求解.但转换得到的初值问题有可能是病态的,因此最后方程组的求解可能会不收敛[7-8].

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单调迭代法[9]通过构造一个单调收敛到问题解的序列来求解,该方法需要知道问题的下界解和上界解,这一般很难找到的.拟线性法[10-11]是牛顿-拉弗森方法的推广,该方法通过近似求解线性问题产生一个二次单调收敛的序列,但像一般牛顿法一样,需要一个较好的初始猜测,不然可能不收敛.广义拟线性法[12]结合拟线性法和上下界解技术,上下界序列都收敛到问题的解,与单调迭代法一样,找上下界解是不方便的.

本文针对一类延迟二阶线性微分方程的两点边值问题构造二阶有限体积法.首先采用均匀网格来剖分区间,利用线性插值离散等方法来构造二阶有限体积格式.之后分析了误差及收敛性,通过离散模估计、分部积分等得出数值解具有步长的二阶收敛性.最后通过数值实验验证了该二阶有限体积法的可行性与有效性.

1　二阶有限体积格式

考虑在区间$[a,b]$上的二阶微分方程第三边值问题:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right)+qu+ru(x-t)=f(x),x\in(a,b),t\geqslant 0;\tag{1}$$

$$-p(a)u'(a)+\alpha_1u(a)=g_1,p(b)u'(b)+\alpha_2u(b)=g_2.\tag{2}$$

当$x\leqslant a+t$时,$u(x-t)=g(x)$,

式中:$p(x),q(x),r(x),f(x)$充分光滑;$p(x)\geqslant p_{\min}>0,\alpha_1\geqslant 0,\alpha_2\geqslant 0$;$\alpha_1,\alpha_2$为常数.$q(x),r(x)$和$\alpha_i$满足2种情形,即$q(x)\geqslant q_{\min}>0,r(x)\geqslant r_{\min}>0,\alpha_1\geqslant 0,\alpha_2\geqslant 0$或$q(x)\geqslant 0,r(x)\geqslant 0,\alpha_1>0,\alpha_2>0$.

考虑到方程(1)中$u(x-t)$部分,因变量x发生了延迟,当$a<x\leqslant a+t$时,就令函数$u(x-t)$为$g(x)$,因此,式(1)等价于

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right)+qu+rg(x)=f(x),x\in(a,a+t];$$

即

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right)+qu=f(x)-rg(x),x\in(a,a+t].$$

当$a+t<x<b$时,(1)式子中$u(x-t)$才开始延时.当$x=a+t$时,若采用非均匀网格来剖分区间,那么在这点所在的体积很难控制,因此在本文中采用均匀网格来剖分.下面就t的取值,来构造2点边值问题式(1)和式(2)的二阶有限体积格式.

当$b-a<t$时,方程(1)即为

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right)+qu=f(x)-rg(x),x\in(a,b).$$

这种情况文献[4]中给出了详细的讨论,在这就不作分析了.

当$0\leqslant t\leqslant b-a$时,对求解区间$I=[a,b]$进行均匀剖分.先取$d_1=|a-t|,d_2=|b-a-t|,h_0$为$d_1$和$d_2$的最大公约数.给定正整数$c$,以步长$h=\dfrac{h_0}{c}$将区间$[a,b]$均匀划分为$n$个小区间,则存在正整数$N_1$,使得$d_1=hN_1$.

设节点为$a=x_0<x_1<\cdots<x_n=b$,单元$I_i=[x_{i-1},x_i]$,且长度为$h=\dfrac{h_0}{c}$.设单元I_i的中点为$x_{i-1/2}$,并记$I_0^*=[x_0,x_{1/2}],I_i^*=[x_{i-1/2},x_{i+1/2}],I_n^*=[x_{n-1/2},x_n]$,则的$I_i^*(i=0,1,\cdots,n)$构成$I_h$的对偶剖分$I_h^*$,$I_i^*$为节点$x_i$对应的控制体积.

记$p_{i-1/2}=p(x_{i-1/2})$,将方程(1)在控制体积$I_i^*(i=0,1,2,\cdots,n)$上进行积分,利用分部积分公式,并考虑到边值条件(2),得到

$$\alpha_1u(x_0)-p_{1/2}u'(x_{1/2})+\int_{x_0}^{x_{1/2}}qu\mathrm{d}x=\int_{x_0}^{x_{1/2}}[f(x)-\mathrm{rg}]\,\mathrm{d}x;\tag{3}$$

$$p_{i-1/2}u'(x_{i-1/2})-p_{i+1/2}u'(x_{i+1/2})+\int_{I_i^*}qu\mathrm{d}x=\int_{I_i^*}[f(x)-rg]\,\mathrm{d}x,(i=1,2,\cdots,N_1-1)\;; \tag{4}$$

$$p_{N_1-1/2}u'(x_{N_1-1/2})-p_{N_1+1/2}u'(x_{N_1+1/2})+\int_{x_{N_1--1/2}}^{x_{N_1+1/2}}qu\mathrm{d}x+\int_{x_0}^{x\ 1/2}r(x-t)u\mathrm{d}x=$$

$$\int_{x_{N_1-1/2}}^{x_{N_1+1/2}}f(x)\,\mathrm{d}x-\int_{x_{N_1-1/2}}^{x_{N_1}}rg\mathrm{d}x; \tag{5}$$

$$p_{i-1/2}u'(x_{i-1/2})-p_{i+1/2}u'(x_{i+1/2})+\int_{x_{i--1/2}}^{x_{i+1/2}}qu\mathrm{d}x+\int_{x_{i-N_1--1/2}}^{x_{i-N_1+1/2}}r(x-t)u\mathrm{d}x=$$

$$\int_{x_{i--1/2}}^{x_{i+1/2}}f(x)\,\mathrm{d}x,(i=N_1+1,N_1+2,\cdots,n-1); \tag{6}$$

$$p_{n-1/2}u'(x_{n-1/2})+\alpha_2 u(x_n)+\int_{I_n^*}qu\mathrm{d}x+\int_{I_{n-N_1}^*}r(x-t)u\mathrm{d}x=\int_{I_n^*}f(x)\,\mathrm{d}x; \tag{7}$$

在式(3)~式(7)中,

$$u(x_i)=u_{i-1/2}+u'_{i-1/2}(x_i-x_{i-1/2})+\frac{1}{2!}u''_{i-1/2}(x_i-x_{i-1/2})^2+\frac{1}{2!}\int_{x_{i-1/2}}^{x_i}u^{(3)}(x)(x_i-x)^2\mathrm{d}x;$$

$$u(x_{i-1})=u_{i-1/2}+u'_{i-1/2}(x_{i-1}-x_{i-1/2})+\frac{1}{2!}u''_{i-1/2}(x_{i-1}-x_{i-1/2})^2+\frac{1}{2!}\int_{x_{i-1/2}}^{x_{i-1}}u^{(3)}(x)(x_{i-1}-x)^2\mathrm{d}x;$$

将上面 2 式相减得

$$u'(x_{i-1/2})=\frac{u(x_i)-u(x_{i-1})}{h}+R_{i-1/2}. \tag{8}$$

式中:

$$R_{i-1/2}=-\frac{1}{2h}\left[\int_{x_{i-1/2}}^{x_i}u^{(3)}(x)(x_i-x)^2\mathrm{d}x+\int_{x_{i-1}}^{x_{i-1/2}}u^{(3)}(x)(x_{i-1}-x)^2\mathrm{d}x\right].$$

在式(3)~式(7)中,对关于 qu 和 $ru(x-t)$ 的积分项进行线性插值离散.对 $u(x)$ 在区间上作线性插值,并积分得

$$\int_{x_{i-1}}^{x_{i-1/2}}u(x)\,\mathrm{d}x\approx\frac{3h}{8}u(x_{i-1})+\frac{h}{8}u(x_i)\overset{\Delta}{=}S_{i,1}(u); \tag{9}$$

$$\int_{x_{i-1/2}}^{x_i}u(x)\,\mathrm{d}x\approx\frac{h}{8}u(x_{i-1})+\frac{3h}{8}u(x_i)\overset{\Delta}{=}S_{i,2}(u). \tag{10}$$

其误差项为

$$R_{i,1}(u)=\int_{x_{i-1}}^{x_{i-1/2}}u(x)-S_{i,1}(u)=$$

$$\frac{1}{2}\int_{x_{i-1}}^{x_{i-1/2}}u^{(2)}(x)(\,{}^{x}_{i-1/2}-x)2\mathrm{d}x-\frac{h}{8}\int_{x_{i-1}}^{x_i}u^{(2)}(x)(x-x_i)\,\mathrm{d}x; \tag{11}$$

$$R_{i,2}(u)=\int_{x_{i-1/2}}^{x_i}u(x)-S_{i,2}(u)=$$

$$\frac{1}{2}\int_{x_{i-1/2}}^{x_i}u^{(2)}(x)(\,{}^{x}_{i-1/2}-x)2\mathrm{d}x-\frac{h}{8}\int_{x_{i-1}}^{x_i}u^{(2)}(x)(x-x_{i-1})\,\mathrm{d}x. \tag{12}$$

记 $q_i=q(x_i)$, 由式(9)~式(10),对于 $i=1,2,\cdots,n-1$, 有

$$\int_{I_i^*}qu\mathrm{d}x\approx\frac{h}{8}q_{i-1}u(x_{i-1})+\frac{6h}{8}q_iu(x_i)+\frac{h}{8}q_{i+1}u(x_{i+1}). \tag{13}$$

$r_i=r(x_i)$, 由式(9)~式(10),对于 $i=N_1+1,N_1+2,\cdots,n-1$, 有

$$\int_{I_i^*}ru(x-t)\,\mathrm{d}x\approx\frac{h}{8}r_{i-1}u(x_{i-1-N_1})+\frac{6h}{8}r_iu(x_{i-N_1})+\frac{h}{8}r_{i+1}u(x_{i+1-N_1}). \tag{14}$$

将式(8)~式(9)($i=1,u\Rightarrow qu,u\Rightarrow ru(x-t)$),式(10)($i=n,u\Rightarrow qu,u\Rightarrow ru(x-t)$)以及式(11)~式(12)代入式(3)~式(7),舍去二阶截断误差项,并记 u_i 为 $u(x_i)$ 的数值解,得到求解延迟微分方程两点边

值问题的有限体积法:

$$\alpha_1 u_0 - \frac{p_{1/2}}{h}(u_1 - u_0) + \frac{3h}{8}q_0 u_0 + \frac{h}{8}q_1 u_1 = \int_{I_0^*}[f(x) - rg(x)]\mathrm{d}x + g_1; \tag{15}$$

$$\frac{p_{i-1/2}}{h}(u_i - u_{i-1}) - \frac{p_{i+1/2}}{h}(u_{i+1} - u_i) + \frac{h}{8}q_{i-1}u_{i-1} + \frac{6h}{8}q_i u_i + \frac{h}{8}q_{i+1}u_{i+1} =$$

$$\int_{I_i^*}[f(x) - rg(x)]\mathrm{d}x, (i = 1, 2, \cdots, N_1 - 1); \tag{16}$$

$$\frac{p_{N_1-1/2}}{h}(u_{N_1} - u_{N_1-1}) - \frac{p_{N_1+1/2}}{h}(u_{N_1+1} - u_{N_1}) + \frac{h}{8}q_{N_1-1}u_{N_1-1} + \frac{6h}{8}q_{N_1}u_{N_1} + \frac{h}{8}q_{N_1+1}u_{N_1+1} +$$

$$\frac{3h}{8}r_{N_1}u_0 + \frac{h}{8}r_{N_1+1}u_1 = \int_{I_{N_1}^*} f(x)\mathrm{d}x - \int_{N_1-1/2}^{N_1} rg\mathrm{d}x; \tag{17}$$

$$\frac{p_{i-1/2}}{h}(u_i - u_{i-1}) - \frac{p_{i+1/2}}{h}(u_{i+1} - u_i) + \frac{h}{8}q_{i-1}u_{i-1} + \frac{6h}{8}q_i u_i + \frac{h}{8}q_{i+1}u_{i+1} + \frac{h}{8}r_{i-1}u_{i-1-N_1} +$$

$$\frac{6h}{8}r_i u_{i-N_1} + \frac{h}{8}r_{i+1}u_{i+1-N_1} = \int_{I_i^*} f(x)\mathrm{d}x (i = N_1 + 1, \cdots, n - 1); \tag{18}$$

$$\frac{p_{n-1/2}}{h}(u_n - u_{n-1}) + \alpha_2 u_n + \frac{h}{8}q_{n-1}u_{n-1} + \frac{3h}{8}q_n u_n + \frac{h}{8}r_{n-1}u_{n-N_1-1} + \frac{3h}{8}r_n u_{n-N_1} =$$

$$\int_{I_n^*} f(x)\mathrm{d}x + g_2. \tag{19}$$

上式中关于 $[u_0, u_1, u_3 \cdots, u_n]^{\mathrm{T}}$ 的线性方程组,其系数矩阵 $\boldsymbol{A}$ 形式如下:

$$\boldsymbol{A} = \begin{bmatrix}
a_{11} & a_{12} & & & & & & & & & & \\
a_{21} & a_{22} & a_{23} & & & & & & & & & \\
 & a_{31} & a_{32} & a_{33} & & & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & & \\
 & & & \ddots & \ddots & \ddots & & & & & & \\
a_{N_11} & a_{N_12} & & & a_{N_1N_1-1} & a_{N_1N_1} & a_{N_1N_1+1} & & & & & \\
a_{N_1+11} & a_{N_1+12} & a_{N_1+13} & & & a_{N_1+1N_1} & a_{N_1+1N_1+1} & a_{N_1+1N_1+2} & & & & \\
 & a_{N_1+22} & a_{N_1+23} & a_{N_1+24} & & & a_{N_1+2N_1+1} & a_{N_1+2N_1+2} & a_{N_1+2N_1+3} & & & \\
 & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \\
 & & & & a_{nn-N_1-1} & u_{nn-N_1} & a_{nn-N_1+1} & & & a_{nn-1} & a_{nn} & a_{nn+1} \\
 & & & & & a_{n+1n-N_1} & a_{n+1n-N_1+1} & & & & a_{nn} & a_{n+1n+1}
\end{bmatrix}.$$

其中只要 h 适当小,该线性代数方程组可求解.

2　误差分析

本节主要研究方程(1)的有限体积格式的误差分析.首先给出了 3 种简单的离散模的估计格式.在 Sobolev 空间里的 s 阶半模和全模分别用 $|\cdot|_s$ 和 $\|\cdot\|_s$ 来表示,e 是剖分区间上的任意网格函数,离散的 L^2 模、H^1 半模与最大模的格式如下:

$$\| e \|_{0,h} = \sqrt{h}\left\{\frac{1}{2}e_0^2 + \sum_{i=1}^{n-1} e_i^2 + \frac{1}{2}e_n^2\right\} 1/2;$$

$$| e |_{1,h} = \frac{1}{\sqrt{h}}\left\{\sum_{i=1}^{n}(e_i - e_{i-1})^2\right\} 1/2; \ \| e \|_{\infty,h} = \max_{1 \leqslant i \leqslant n} | e_i |.$$

引理 1 假设 I_h 是$[a,b]$区间上的均匀剖分,对上面任意网格函数 $\{e_i\}_{i=0}^{n}$,有

$$\|e\|_{0,h}^{2} \leqslant (b-a)(e_0^2+e_n^2)+\frac{(b-a)^2}{2}|e|_{1,h}^{2};$$

$$\|e\|_{\infty,h}^{2} \leqslant 2\max\{e_0^2,e_n^2\}+\frac{b-a}{2}|e|_{1,h}^{2}; \tag{20}$$

$$\|e\|_{\infty,h} \leqslant \frac{1}{\sqrt{b-a}}\|e\|_{0,h}+\sqrt{b-a}\,|e|_{1,\mathrm{h}}.$$

证 当 $1 \leqslant i \leqslant n-1$ 时,

$$e_i = e_0+\sum_{j=1}^{i}(e_j-e_{j-1});e_i=e_n-\sum_{j=i+1}^{n}(e_j-e_{j-1}).$$

上 2 式 2 边同时平方得

$$e_i^2=\left[e_0+\sum_{j=1}^{i}(e_j-e_{j-1})\right]^2;e_i^2=\left[e_n-\sum_{j=i+1}^{n}(e_j-e_{j-1})\right]^2.$$

利用 Cauchy-Schwarz 不等式,则有

$$e_i^2 \leqslant 2\left[e_0^2+(x_i-a)\frac{1}{h}\sum_{j=1}^{i}(e_j-e_{j-1})^2h\right];e_i^2 \leqslant 2\left[e_n^2+(b-x_i)\frac{1}{h}\sum_{j=i+1}^{n}(e_j-e_{j-1})^2h\right].$$

将上 2 式分别乘以 $(b-x_i)$,(x_i-a) 得

$$(b-x_i)e_i^2 \leqslant 2(b-x_i)\left[e_0^2+(x_i-a)\frac{1}{h}\sum_{j=1}^{i}(e_j-e_{j-1})^2h\right];(x_i-a)e_i^2 \leqslant 2(x_i-a)\left[e_n^2+(b-x_i)\frac{1}{h}\sum_{j=i+1}^{n}(e_j-e_{j-1})^2h\right].$$

将上 2 式相加得

$$(b-a)e_i^2 \leqslant 2(b-x_i)e_0^2+2(x_i-a)e_n^2+2(x_i-a)(b-x_i)|e|_{1,h}^2. \tag{21}$$

从而证明了式(20)中第 2 个不等式.

将式(21)式 2 边同时乘以 h, 将 i 从 1 到 n 求和,同时式(21)2 边同时乘以 h, 将 i 从 0 到 $n-1$ 求和,将新得到的 2 个式子相加可以得到

$$2(b-a)\|e\|_{0,h}^2 \leqslant 2h|e|_{1,h}^2\left[\sum_{i=1}^{n}(x_i-a)(b-x_i)+\sum_{i=0}^{n-1}(x_i-a)(b-x_i)\right]+ 2he_0^2\left[\sum_{i=1}^{n}(b-x_i)+\sum_{i=0}^{n-1}(b-x_i)\right]+2he_n^2\left[\sum_{i=1}^{n}(x_i-a)+\sum_{i=0}^{n-1}(x_i-a)\right]. \tag{22}$$

则有

$$h\sum_{i=1}^{n}(b-x_i)+h\sum_{i=0}^{n-1}(b-x_i)=\sum_{i=1}^{n}(2b-x_i-x_{i-1})(x_i-x_{i-1})= 2b(b-a)-\sum_{i=1}^{n}(x_i^2-x_{i-1}^2)=(b-a)^2.$$

同理可得

$$\sum_{i=1}^{n}(x_i-a)h+\sum_{i=0}^{n-1}(x_i-b)h=(b-a)^2.$$

因为

$$\max_{x\in[a,b]}(x-a)(b-x)=\frac{(b-a)^2}{4}.$$

将上面的式子代入式(22),就证明了式(20)中的第一个不等式成立.

设 $e_{i_0}=\min\limits_{0\leqslant i\leqslant n}|e_i|$,则有

$$|e_{i_0}| \leqslant \frac{1}{b-a}\left(\frac{h_1}{2}|e_0| + \sum_{j=1}^{n-1}\frac{h_j + h_{j+1}}{2}|e_j| + \frac{h_n}{2}|e_n|\right).$$

令

$$e_i = \begin{cases} e_{i_0} + \sum\limits_{j=i_0+1}^{i}\delta_x^- e_j h_j, i \geqslant i_0; \\ e_{i_0} - \sum\limits_{j=i+1}^{i_0}\delta_x^- e_j h_j, i \leqslant i_0. \end{cases}$$

由 Cauchy 不等式,得到

$$|e_i| \leqslant \frac{1}{b-a}\left(\frac{h}{2}|e_0| + \sum_{j=1}^{n-1} h|e_j| + \frac{h}{2}|e_n|\right) + \sum_{j=1}^{n}|\delta_x^- e_j| h \leqslant$$

$$\frac{1}{\sqrt{b-a}}\|e\|_{0,h} + \sqrt{b-a}\;|e|_{1,h}.$$

在第 1 节中式(15)~式(19)为求解延迟微分方程两点边值问题的有限体积方法,对该方法进行误差分析,采用离散模.设 $e_i = u(x_i) - u_i$, 由式(15)~式(19)可得到误差方程

$$\alpha_1 e_0 - \frac{p_{1/2}}{h}(e_1 - e_0) + \frac{3h}{8}q_0 e_0 + \frac{h}{8}q_1 e_1 = p_{1/2}R_{1/2} - R_{1,1}(qu); \tag{23}$$

$$\frac{p_{i-1/2}}{h}(e_i - e_{i-1}) - \frac{p_{i+1/2}}{h}(e_{i+1} - e_i) + \frac{h}{8}q_{i-1}e_{i-1} + \frac{6h}{8}q_i e_i + \frac{h}{8}q_{i+1}e_{i+1} = p_{i+1/2}R_{i+1/2} -$$
$$p_{i-1/2}R_{i-1/2} - R_{i+1,1}(qu) - R_{i,2}(qu), (1 \leqslant i < N_1); \tag{24}$$

$$\frac{p_{N_1-1/2}}{h}(e_{N_1} - e_{N_1-1}) - \frac{p_{N_1+1/2}}{h}(e_{N_1+1} - e_{N_1}) + \frac{h}{8}q_{N_1-1}e_{N_1-1} + \frac{6h}{8}q_{N_1}e_{N_1} + \frac{h}{8}q_{N_1+1}e_{N_1+1} + \frac{3h}{8}r_{N_1}e_0 +$$
$$\frac{h}{8}r_{N_1+1}e_1 = p_{N_1+1/2}R_{N_1+1/2} - p_{N_1-1/2}R_{N_1-1/2} - R_{N_1+1,1}(qu) - R_{N_1,2}(qu) - R_{N_1+1,1}[ru(x-t)]; \tag{25}$$

$$\frac{p_{i-1/2}}{h}(e_i - e_{i-1}) - \frac{p_{i+1/2}}{h}(e_i - e_{i-1}) + \frac{h}{8}q_{i-1}e_{i-1} + \frac{6h}{8}q_i e_i + \frac{h}{8}q_{i+1}e_{i+1} +$$
$$\frac{h}{8}r_{i-1}e_{i-1-N_1} + \frac{6h}{8}r_i e_{i-N_1} + \frac{h}{8}r_{i+1}e_{i+1-N_1} = p_{i+1/2}R_{i+1/2} - p_{i-1/2}R_{i-1/2} -$$
$$R_{i+1,1}(qu) - R_{i,2}(qu) - R_{i+1,1}[ru(x-t)] - R_{i,2}[ru(x-t)], (N_1 + 1 \leqslant i \leqslant n-1); \tag{26}$$

$$\frac{p_{n-1/2}}{h}(e_n - e_{n-1}) - \alpha_2 e_n + \frac{h}{8}q_{n-1}e_{n-1} + \frac{3h}{8}q_n e_n + \frac{h}{8}r_{n-1}e_{n-1-N_1} + \frac{3h}{8}r_n e_{n-N_1} =$$
$$-p_{n-1/2}R_{n-1/2} - R_{n,2}(qu) - R_{n,2}[ru(x-t)]. \tag{27}$$

将式(23)~式(27)分别乘以 $e_0, e_i, e_{N_1}, e_i, e_n$, 并将 i 从 0 到 n 相加得到

$$L_1 = -p_{1/2}\frac{e_1 - e_0}{h}e_0 + \sum_{i=1}^{n}\left(p_{i-1/2}\frac{e_i - e_{i-1}}{h} - p_{i+1/2}\frac{e_i - e_{i-1}}{h}\right)e_i + p_{n-1/2}\frac{e_n - e_{n-1}}{h}e_n;$$

$$L_2 = \frac{h}{8}(3q_0e_0 + q_1e_1)e_0 + \sum_{i=1}^{n-1}\left(\frac{h}{8}q_{i-1}e_{i-1} + \frac{6h}{8}q_i e_i + \frac{h}{8}q_{i+1}e_{i+1}\right)e_i + \frac{h}{8}(q_{n-1}e_{n-1} + 3q_n e_n)e_n;$$

$$L_3 = \frac{h}{8}(3r_{N_1}e_0 + r_{N_1+1}e_1)e_{N_1} + \sum_{i=N_1+1}^{n-1}\left(\frac{h}{8}r_{i-1}e_{i-N_1-1} + \frac{6h}{8}r_i e_{i-N_1} + \frac{h}{8}r_{i+1}e_{i-N_1+1}\right)e_i +$$
$$\frac{h}{8}(r_{n-1}e_{n-N_1-1} + 3r_n e_{n-N_1})e_n;$$

$$L_4 = \alpha_1 e_0^2 + \alpha_2 e_n^2;$$

$$T_1 = p_{1/2}R_{1/2}e_0 + \sum_{i=1}^{n-1}(p_{i+1/2}R_{i+1/2} - p_{i-1/2}R_{i-1/2})e_i - p_{n-1/2}R_{n-1/2}e_n;$$

$$T_2 = -R_{1,1}(qu)e_0 - \sum_{i=1}^{n-1}(R_{i+1,1}(qu) + R_{i,2}(qu))e_i - R_{n,2}(qu)e_n;$$

$$T_3 = -R_{N_1+1,1}[ru(x-t)]e_{N_1} - \sum_{i=N_1+1}^{n-1}\{R_{i+1,1}[ru(x-t)] + R_{i,2}[ru(x-t)]\}e_i - R_{n,2}[ru(x-t)]e_n.$$ 则式(23)~式(27)可以表示为 $L_1 + L_2 + L_3 + L_4 = T_1 + T_2 + T_3$.

对 L_1 采用分部求和,得

$$L_1 = \sum_{i=1}^{n} p_{i-1/2}\left(\frac{(e_i - e_{i-1})^2}{h}\right) \geqslant p_{\min} |e|_{1,h}^2.$$

可以将 L_2 写成 $L_2 = L_{21} + L_{22}$, 其中

$$L_{21} = \frac{h}{2}q_0e_0^2 + h\sum_{i=1}^{n-1}q_ie_i^2 + \frac{h}{2}q_ne_n^2;$$

$$L_{22} = \frac{h}{8}(q_1e_1 - q_0e_0)e_0 + \frac{h}{4}\sum_{i=1}^{n-1}[-(q_ie_i - q_{i-1}e_{i-1}) + (q_{i+1}e_{i+1} - q_ie_i)]e_i - \frac{h}{8}(q_ne_n - q_{n-1}e_{n-1})e_n = -\frac{h}{8}\sum_{i=1}^{n}(q_ie_i - q_{i-1}e_{i-1})(e_i - e_{i-1})$$

由 Cauchy 不等式,得

$$|L_{22}| \leqslant \sum_{i=1}^{n}\frac{h^2}{16}(q_ie_i - q_{i-1}e_{i-1})^2 + \frac{h}{16}|e|_{1,h}^2 \leqslant \frac{h^2}{8}q_0^2e_0^2 + \frac{h^2}{4}\sum_{i=1}^{n}q_i^2e_i^2 + \frac{h^2}{8}q_n^2e_n^2 + \frac{h}{16}|e|_{1,h}^2. \tag{28}$$

将 L_3 写成 $L_3 = L_{31} + L_{32}$, 式中:

$$L_{31} = \frac{h}{2}r_{N_1}e_0e_{N_1} + h\sum_{n=N_1+1}^{n-1}r_ie_{i-N_1}e_i + \frac{h}{2}r_ne_{n-N_1}e_n;$$

$$L_{32} = -\frac{h}{8}\sum_{i=N_1+1}^{n}(r_ie_{i-N_1} - r_{i-1}e_{i-1-N_1})(e_i - e_{i-1}).$$

利用 Cauchy 不等式得

$$|\mathrm{L}_{32}| \leqslant \frac{h^2}{8}r_{N_1}^2e_0^2 + \frac{h^2}{4}\sum_{i=N_1+1}^{n-1}r_i^2e_{i-N_1}^2 + \frac{h^2}{8}r_n^2e_{n-N_1}^2 + \frac{h}{16}\sum_{i=1}^{N_1}(e_i - e_{i-1})^2 \leqslant \frac{h^2}{8}r_{N_1}^2e_0^2 + \frac{h^2}{4}\sum_{i=N_1+1}^{n-1}r_i^2e_{i-N_1}^2 + \frac{h^2}{8}r_n^2e_{n-N_1}^2 + \frac{h}{16}|e|_{1,h}^2. \tag{29}$$

由 L_1,L_2 和 L_3 的估计,得

$$L_1 + L_2 + L_3 + L_4 \geqslant \left(p_{\min} - \frac{h}{8}\right)|e|_{1,h}^2 + \left(1 - \frac{hq_0}{4}\right)\frac{h}{2}q_0e_0^2 + \sum_{i=1}^{n-1}\left(1 - \frac{hq_i}{4}\right)hq_ie_i^2 + \left(1 - \frac{hq_n}{4}\right)\frac{h}{2}q_ne_n^2 + \left(\frac{e_{N_1}}{e_0} - \frac{hr_{N_1}}{4}\right)\frac{h}{2}r_{N_1}e_0^2 + \sum_{i=N_1+1}^{n-1}\left(\frac{e_i}{e_{i-N_1}} - \frac{hr_i}{4}\right)hr_ie_{i-N_1}^2 + \left(1 - \frac{hr_n}{4}\right)\frac{h}{2}r_ne_{n-N_1}^2 + \alpha_1e_0^2 + \alpha_2e_n^2. \tag{30}$$

接着对 T_1,T_2 和 T_3 进行估计,利用分部求和公式化简得

$$T_1 = -\sum_{i=1}^{n}p_{i-1/2}R_{i-1/2}(e_i - e_{i-1}).$$

由 ε - Cauchy 不等式得:

$$|T_1| \leqslant \frac{h}{4\varepsilon_1}\sum_{i=1}^{n}(p_{i-1/2}R_{i-1/2})^2 + \varepsilon_1|e|_{1,h}^2.$$

在式(8)中定义 $R_{i-1/2}$ 得:

$$(p_{i-1/2}R_{i-1/2})^2 \leqslant \frac{p_{i-1/2}^2}{2h^2}\left\{\left[\int_{x_{i-1/2}}^{x_i} u^{(3)}(x)(x_i-x)^2\mathrm{d}x\right]2+\left[\int_{x_{i-1}}^{x_{i-1/2}} u^{(3)}(x)(x_{i-1}-x)^2\mathrm{d}x\right]2\right\}.$$

令 $p_{\max}=\max\limits_{x\in[a,b]}p(x)$, 采用 *Hölder* 不等式,并整理可得

$$\sum_{i=1}^{n}(p_{i-1/2}R_{i-1/2})^2 h \leqslant \frac{p_{\max}^2}{320}\sum_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x.$$

从而 $|T_1| \leqslant \dfrac{p_{\max}^2}{4\varepsilon_1\times 320}\sum\limits_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x+\varepsilon_1|e|_{1,h}^2.$　　(31)

对 T_2,T_3 利用分部求和公式,有

$$T_2=-\sum_{i=1}^{n}R_{i,1}(qu)e_{i-1}-\sum_{i=1}^{n}R_{i,2}(qu)e_i;$$

$$T_3=-\sum_{i=N_1+1}^{n}R_{i,1}[ru(x-t)]e_{i-1}-\sum_{i=N_1+1}^{n}R_{i,2}[ru(x-t)]e_i.$$

在式(11)~式(12)中对 $R_{i,1}(qu),R_{i,2}(qu)$ 定义及 Holder 不等式,整理得

$$|T_2| \leqslant \frac{h^4}{8\varepsilon_2\times 320}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(qu)^{(2)}(x)^2\mathrm{d}x+4\varepsilon_2\|e\|_{0,h}^2; \tag{32}$$

$$|T_3| \leqslant \frac{h^4}{8\varepsilon_3\times 320}\sum_{i=N_1+1}^{n}\int_{x_{i-1}}^{x_i}(qu)^{(2)}(x)\mathrm{d}x+2\varepsilon_3\sum_{i=N_1+1}^{n}he_{i-1}^2+2\varepsilon_3\sum_{i=N_1+1}^{n}he_i^2.$$

设存在 N_1, 使得 $\|e\|_{0,h}=\left\{\dfrac{h}{2}e_{N_1}^2+h\sum\limits_{i=N_1}^{n-1}e_i^2+\dfrac{h}{2}e_n^2\right\}^{1/2}$, 则可以得到

$$|T_3| \leqslant \frac{h^4}{8\varepsilon_3\times 320}\sum_{i=N_1+1}^{n}\int_{x_{i-1}}^{x_i}(ru)^{(2)}(x-t)^2\mathrm{d}x+4\varepsilon_3\|e\|_{0,h}^2. \tag{33}$$

由式(28)~式(33)中,对 L_1,L_2,L_3,T_1,T_2 和 T_3 的估计,得

$$(p_{\min}-\frac{h}{8})|e|_{1,h}^2+(1-\frac{hq_0}{4})\frac{h}{2}q_0e_0^2+\sum_{i=1}^{n-1}(1-\frac{hq_i}{4})hq_ie_i^2+(1-\frac{hq_n}{4})\frac{h}{2}q_ne_n^2+$$

$$(\frac{e_{N_1}}{e_0}-\frac{hr_{N_1}}{4})\frac{h}{2}r_{N_1}e_0^2+\sum_{i=N_1+1}^{n-1}(\frac{e_i}{e_{i-N_1}}-\frac{hr_i}{4})hr_ie_{i-N_1}^2+(\frac{e_n}{e_{n-N_1}}-\frac{hr_n}{4})\frac{h}{2}r_ne_{n-N_1}^2+$$

$$\alpha_1e_0^2+\alpha_2e_n^2 \leqslant \frac{p_{\max}^2}{4\varepsilon_1\times 320}\sum_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x+$$

$$\frac{h^4}{8\varepsilon_2\times 320}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(qu)^{(2)}(x)^2\mathrm{d}x+\frac{h^4}{8\varepsilon_3\times 320}\sum_{i=N_1+1}^{n}\int_{x_{i-1}}^{x_i}(ru)^{(2)}(x-t)\mathrm{d}x+$$

$$\varepsilon_1|e|_{1,h}^2+4\varepsilon_2\|e\|_{0,h}^2+4\varepsilon_3\|e\|_{0,h}^2. \tag{34}$$

当 $q(x)\geqslant q_{\min}>0,r(x)\geqslant r_{\min}>0,\alpha_1\geqslant 0,\alpha_2\geqslant 0$ 时,取, $\varepsilon_1=p_{\min}/2,\varepsilon_2=q_{\min}/8$, $\varepsilon_3=\dfrac{3}{16}r_{\min}e_r,e_{r_i}=\dfrac{e_i}{e_{i-N_1}},e_r=\max\left\{\left|\dfrac{e_i}{e_{i-N_1}}\right|,1\right\}(i=N_1,N_1+1,\cdots,n)$, 存在 $c_0>c>0$,

使得 $h_1=\dfrac{b-a}{c_0h_0},h_1r_i\leqslant e_r(i=N_1,N_1+1,\cdots,n)$, 得

$$(p_{\min}-\frac{h_1}{8})|e|_{1,h}^2+(1-\frac{h_1q_0}{4})\frac{h}{2}q_0e_0^2+\sum_{i=1}^{n-1}(1-\frac{h_1q_i}{4})hq_ie_i^2+(1-\frac{h_1q_n}{4})\frac{h_1}{2}q_ne_n^2-$$

$$\frac{5e_r}{4}\left(\frac{h_1}{2}r_{N_1}e_0^2+\sum_{i=N_1+1}^{n-1}h_1r_ie_{i-N_1}^2+\frac{h}{2}r_ne_{n-N_1}^2\right)+\alpha_1e_0^2+\alpha_2e_n^2 \leqslant$$

$$\frac{p_{\max}^2}{4\varepsilon_1\times 320}\sum_{i=1}^{n}h_1^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x+\frac{h_1^4}{8\varepsilon_2\times 320}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(qu)^{(2)}(x)^2\mathrm{d}x+$$

$$\frac{h_1^4}{8\varepsilon_3 \times 320}\sum_{i=N_1+1}^{n}\int_{x_{i-1}}^{x_i}(ru)^{(2)}(x-t)\mathrm{d}x + \varepsilon_1 |e|_{1,h}^2 + 4\varepsilon_2 \|e\|_{0,h}^2 + 4\varepsilon_3 \|e\|_{0,h}^2.$$

从而得出

$$(\frac{p_{\min}}{2} - \frac{h_1}{8})|e|_{1,h}^2 + (\frac{1}{2} - \frac{h_1 q_0}{4})\frac{h}{2}q_0 e_0^2 + \sum_{i=1}^{n-1}(\frac{1}{2} - \frac{h_1 q_i}{4})hq_i e_i^2 + (\frac{1}{2} - \frac{h_1 q_n}{4})\frac{h}{2}q_n e_n^2 -$$

$$2r_{\min}e_r \|e\|_{0,h}^2 \leqslant \frac{p_{\max}^2}{4\varepsilon_1 \times 320}\sum_{i=1}^{n}h_1^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x + \frac{h_1^4}{8\varepsilon_2 \times 320}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(qu)^{(2)}(x)^2\mathrm{d}x +$$

$$\frac{h_1^4}{8\varepsilon_3 \times 320}\sum_{i=N_1+1}^{n}\int_{x_{i-1}}^{x_i}(ru)^{(2)}(x-t)\mathrm{d}x.$$

存在 $c_1 \geq c_0 > c > 0$, 使得 $h_2 = \dfrac{b-a}{c_1 h_0}, h_2 r_i \leqslant 1(i = 1,2,\cdots,n)$,则有

$$(\frac{p_{\min}}{2} - \frac{h_2}{8})|e|_{1,h}^2 + \frac{q_{\min}}{4}\|e\|_{0,h}^2 + (\frac{r_0}{4} - |r_0|)r_{\min}\|e\|_{0,h}^2 \leqslant$$

$$\frac{p_{\max}^2}{640p_{\min}}\sum_{i=1}^{n}h_2^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x + \frac{h_2^4}{320q_{\min}}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(qu)^{(2)}(x)^2\mathrm{d}x +$$

$$\frac{h_2^4}{480r_0 r_{\min}}\sum_{i=N_1+1}^{n}\int_{x_{i-1}}^{x_i}(ru)^{(2)}(x-t)^2\mathrm{d}x. \tag{35}$$

取 $h = \min\{h_1, h_2\}$,则有

$$(\frac{p_{\min}}{2} - \frac{h}{8})|e|_{1,h}^2 + (\frac{q_{\min}}{4} - 2r_{\min}e_r)\|e\|_{0,h}^2 \leqslant$$

$$\frac{p_{\max}^2}{640p_{\min}}|u|_3^2 + \frac{h^4}{320q_{\min}}|qu|_2^2 + \frac{h^4}{480r_0 r_{\min}}|ru(x-t)|_2^2. \tag{36}$$

当 $q(x) \geq 0, r(x) \geq 0, \alpha_1 > 0, \alpha_2 > 0$, 存在 $c_2 \geq c > 0, c_3 \geq c > 0, c_4 = \max\{c_2, c_3\}$,

$h = \dfrac{b-a}{c_4 h_0}$, 使得 $[1 - hq(x)] \geq 0, (\dfrac{e_i}{e_{i-N_1}} - \dfrac{hr(x)}{4}) \geq 0(i = N_1 \cdots n)$, 则由式(33)和引理 1 得

$$(p_{\min} - \frac{h}{16})|e|_{1,h}^2 + \alpha_1 e_0^2 + \alpha_2 e_n^2 \leqslant \frac{p_{\max}^2}{4\varepsilon_1 \times 320}\sum_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x +$$

$$\frac{1}{8\varepsilon_2 \times 320}\sum_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}(qu)(3)(x)^2\mathrm{d}x + \frac{1}{8\varepsilon_3 \times 320}\sum_{i=N_1}^{n}h^4\int_{x_{i-1}}^{x_i}(ru)(3)(x-t)^2\mathrm{d}x +$$

$$\varepsilon_1 |e|_{1,h}^2 + 4\varepsilon_2(b-a)(e_0^2 + e_n^2) + 2\varepsilon_2(b-a)^2|e|_{1,h}^2 +$$

$$4\varepsilon_3(b-a-t)(e_0^2 + e_n^2) + 2\varepsilon_2(b-a-t)^2|e|_{1,h}^2. \tag{37}$$

取

$$\varepsilon_1 = p_{\min}/4, \varepsilon_2 = \min(\frac{p_{\min}}{16(b-a)^2}, \frac{\alpha_1}{16(b-a)}, \frac{\alpha_2}{16(b-a)});$$

$$\varepsilon_3 = \min(\frac{p_{\min}}{16(b-a-t)^2}, \frac{\alpha_1}{16(b-a-t)}, \frac{\alpha_2}{16(b-a-t)}).$$

则有,

$$(\frac{p_{\min}}{2} - \frac{h}{16})|e|_{1,h}^2 + \frac{\alpha_1}{2}e_0^2 + \frac{\alpha_2}{2}e_n^2 \leqslant \frac{p_{\max}^2}{320p_{\min}}\sum_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}u^{(3)}(x)^2\mathrm{d}x +$$

$$\frac{1}{8\varepsilon_2 \times 320}\sum_{i=1}^{n}h^4\int_{x_{i-1}}^{x_i}(qu)^{(3)}(x)^2\mathrm{d}x + \frac{1}{8\varepsilon_3 \times 320}\sum_{i=N_1}^{n}h^4\int_{x_{i-1}}^{x_i}(ru)^{(3)}(x-t)^2. \tag{38}$$

可得到

$$(\frac{p_{\min}}{4} - \frac{h}{16})|e|_{1,h}^2 + \frac{\alpha_1}{2}e_0^2 + \frac{\alpha_2}{2}e_n^2 \leqslant$$

$$\frac{p_{\max}^2}{320p_{\min}}|u|_3^2+\frac{h^4}{2\,560\varepsilon_2}|qu|_2^2+\frac{h^4}{2\,560\varepsilon_3}|ru(x-t)|_2^2. \tag{39}$$

由式(36)~式(39)和引理 1，可得出以下定理 1.

定理 1　假设 $p(x)\in C^1(I)$，$q(x)\geq q_{\min}>0$，$r(x)\geq r_{\min}>0$，$\alpha_1\geq 0$，$\alpha_1\geq 0$ 或 $q(x)\geq 0$，$r(x)\geq 0$，$\alpha_1>0$，$\alpha_1>0$，$u(x)\in H^3(I)$ 是式(1)和式(2)的解，求解延迟微分方程两点边值问题的有限体积法式(15)~式(19)在点 x_i 处的数值解为 u_i，u_i 按照离散 H^1 半范数，L^2 范数二阶收敛于 $u(x_i)$，且误差 $e_i=u(x_i)-u_i$，有如下估算模式

$$|e|_{1,h}\leqslant Ch^2\ \|u\|_3,\ \|e\|_{0,h}\leqslant Ch^2\ \|u\|_3,\ \|e\|_{\infty,h}\leqslant Ch^2\ \|u\|_3.$$

3　数值实验

限于篇幅，本节给出一个算例. 线性方程组(15)~(19)采用高斯消元法，使用 Matlab 编程. 分别用 eh1，eh2 和 Max_e 表示数值解误差的 H^1 半模，L^2 模与最大模.

式(1)~式(2)中的各参数为，$\alpha_1=g_1=\alpha_2=1$，$g_2=e$，$a=0$，$b=1$，$t=1/2$，$p=r=1$，$q=2$，$g(x)=\frac{e^{x-1/2}-e^{-x+1/2}}{2}$，$f(x)=\frac{e^x+e^{-x}}{2}+\frac{e^{x-1/2}-e^{-x+1/2}}{2}$，其精确解 $u(x)=\frac{e^x+e^{-x}}{2}$.

分别取 $n=2,4,8,10$ 时的数值解如下图 1 所示. 从图 1 可以看出，随着网格节点的增加，数值解与精确解的误差越来越小. 当 $n=2$ 时，误差比较大，当 $n=4$ 时，误差明显减少，当 n 增加到 8 或者 10 时，数值解与精确解没有明显差异.

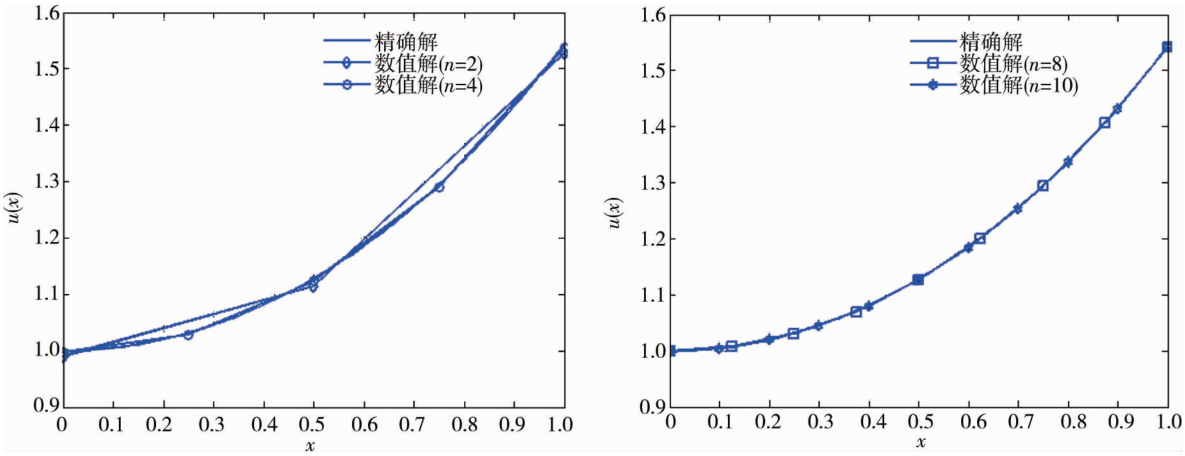

图 1　算例的精确解，数值解($n=2,4,8,10$)

为了进一步验证算法的收敛性，表 1 进一步比较 $n=10$ 和 100 时的 3 种数值误差. 从表 1 可以看出，数值解按照 H^1 半模，L^2 模、最大模几乎是二阶收敛的.

表 1　$n=10,100$ 的数值解误差

n	误差		
	Max_e	eh2	eh1
10	5.656 2E-004	5.099 1E-004	2.162 4E-004
100	5.656 5E-006	5.100 8E-006	2.168 5E-006

4　结论

1)有限体积法是求解延迟二阶微分方程两点边值问题的有效数值方法，不但简单容易实现，而且具有二阶精度.

2)误差分析和数值结果都表明该方法在 H^1 半范数、L^2 范数以及最大范数等多种范数下都是二阶收

敛的.

3)方法的构造和误差分析的技巧具有一定的普适性.

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